Symbiotic Symmetries of the Two-Higgs-Doublet Model

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The new phenomenon of *symbiotic symmetries* is described in the context of the Two-Higgs-Doublet Model (THDM). The quartic potential has two or more separate sectors with unequal symmetries, but these unequal symmetries persist even though the different sectors are renormalized by one another. We discuss all such symmetries of the THDM, consistent with the $SU(2) \times U(1)$ gauge interactions, using the Pauli formalism.

Much attention has been paid to the Two-Higgs-Doublet Model (THDM) which has a long history, with applications to many diverse issues in high-energy physics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. One important reason is supersymmetry, where the minimal extension of the Standard Model (SM) of quarks and leptons requires two Higgs doublet superfields. On the other hand, even without supersymmetry, THDM's have interesting properties in their own right. For example, CP violation is allowed in the Higgs sector itself, and there are five physical Higgs bosons to be searched for at the upcoming Large Hadron Collider (LHC), among many other possible considerations.

The most general THDM Higgs potential, which obeys electroweak gauge symmetry and is renormalizable, has three quadratic mass terms and seven quartic coupling terms. Experimentally, the non-observation of flavor-changing neutral currents limits the available parameter space, and may correspond to a symmetry [21, 22, 23, 24, 25]. If some version of the THDM is realized in Nature, it will be important to understand why certain terms of the general THDM should be suppressed. An underlying symmetry principle may be responsible for this selection, which must of course be also phenomenologically acceptable.

In this work we look systematically for symmetries [3, 26] in the most general THDM Higgs potential which are preserved by the renormalization-group equations (rge's) in the presence of gauge interactions. We make use of the powerful new formalism recently proposed to describe the general THDM in a concise way [6, 11, 12, 16]. In this formalism, all gauge-invariant expressions are given in terms of four real *gauge-invariant functions*. In particular all quartic couplings are incorporated into one real, symmetric 4×4 matrix. As we will show, in terms of the rge's of this quartic coupling matrix, symmetries of the THDM Higgs potential become very transparent. Our key find is that there are cases in which two or three separate groups of terms have unequal symmetries and yet each retains its form even after renormalization. We call this the phenomenon of *symbiotic symmetries*.

In order to make this article self-contained, we review here briefly the usage of *gauge-invariant functions*. Consider first the most general potential of two Higgs doublets Φ_1, Φ_2 in the conventional notation [4]

$$
\begin{aligned}
V = {}& m_{11}^2 (\Phi_1^\dagger \Phi_1) + m_{22}^2 (\Phi_2^\dagger \Phi_2) - m_{12}^2 (\Phi_1^\dagger \Phi_2) - (m_{12}^2)^* (\Phi_2^\dagger \Phi_1) \\
& + \frac{1}{2} \lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2} \lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) \\
& + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \frac{1}{2} [\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \lambda_5^* (\Phi_2^\dagger \Phi_1)^2] \\
& + [\lambda_6 (\Phi_1^\dagger \Phi_2) + \lambda_6^* (\Phi_2^\dagger \Phi_1)](\Phi_1^\dagger \Phi_1) + [\lambda_7 (\Phi_1^\dagger \Phi_2) + \lambda_7^* (\Phi_2^\dagger \Phi_1)](\Phi_2^\dagger \Phi_2).
\end{aligned}
\tag{1}
$$

Hermiticity of the Lagrangian requires the parameters m_{12}^2, $\lambda_{5,6,7}$ to be complex and all other parameters to be real. Owing to $SU(2)_L \times U(1)_Y$ gauge invariance, only terms of the form $(\Phi_i^\dagger \Phi_j)$ with $i, j = 1, 2$ may occur in the Higgs potential. The Hermitian, positive semi-definite 2×2 matrix of all possible scalar products of this form may be decomposed in the following way [11, 12],

$$
\underline{K} := \begin{pmatrix} \Phi_1^\dagger \Phi_1 & \Phi_2^\dagger \Phi_1 \\ \Phi_1^\dagger \Phi_2 & \Phi_2^\dagger \Phi_2 \end{pmatrix} = \frac{1}{2} (K_0 \mathbf{1}_2 + K_i \sigma_i),
\tag{2}
$$

with Pauli matrices σ_i, $i = 1, 2, 3$, and the convention of summing over repeated indices is adopted. Specifically, these real gauge-invariant functions are defined as

$$
K_0 = \Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2, \quad K_1 = \Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1, \quad K_2 = i\Phi_2^\dagger \Phi_1 - i\Phi_1^\dagger \Phi_2, \quad K_3 = \Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2.
\tag{3}
$$

The matrix $\underline{K}$ in (2) is positive semi-definite with two conditions for the gauge-invariant functions:

$$K_0 \geq 0, \qquad K_\alpha K_\alpha \equiv K_0^2 - K_1^2 - K_2^2 - K_3^2 \geq 0. \tag{4}$$

For convenience, we introduce the shorthand vector notation $\mathbf{K} = (K_1, K_2, K_3)^{\mathrm{T}}$. For any K_0 and $\mathbf{K}$, it is possible to find doublet fields $\Phi_{1,2}$ obeying (3). These doublets then form an gauge orbit. In terms of the gauge-invariant functions, the general THDM potential may be written in the simple form

$$V = V_2 + V_4, \qquad \text{with } V_2 = \xi_\alpha K_\alpha, \quad V_4 = \eta_{\alpha\beta} K_\alpha K_\beta, \tag{5}$$

where ξ_α is a real 4-vector and $\eta_{\alpha\beta}$ is a real, symmetric 4×4 matrix. Expressed in terms of the conventional parameters, these tensors read

$$\xi_\alpha = \frac{1}{2} \left(m_{11}^2 + m_{22}^2, \ -2\,\mathrm{Re}(m_{12}^2), \ 2\,\mathrm{Im}(m_{12}^2), \ m_{11}^2 - m_{22}^2 \right) \tag{6}$$

and

$$\eta_{\alpha\beta} = \frac{1}{4} \begin{pmatrix} \frac{1}{2}(\lambda_1 + \lambda_2) + \lambda_3 & \mathrm{Re}(\lambda_6 + \lambda_7) & -\mathrm{Im}(\lambda_6 + \lambda_7) & \frac{1}{2}(\lambda_1 - \lambda_2) \\ \mathrm{Re}(\lambda_6 + \lambda_7) & \lambda_4 + \mathrm{Re}(\lambda_5) & -\mathrm{Im}(\lambda_5) & \mathrm{Re}(\lambda_6 - \lambda_7) \\ -\mathrm{Im}(\lambda_6 + \lambda_7) & -\mathrm{Im}(\lambda_5) & \lambda_4 - \mathrm{Re}(\lambda_5) & -\mathrm{Im}(\lambda_6 - \lambda_7) \\ \frac{1}{2}(\lambda_1 - \lambda_2) & \mathrm{Re}(\lambda_6 - \lambda_7) & -\mathrm{Im}(\lambda_6 - \lambda_7) & \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 \end{pmatrix}. \tag{7}$$

It was shown that the formalism of gauge-invariant functions is advantageous in describing THDM's. That is, conditions for stability, stationarity, electroweak symmetry breaking, and CP violation of any THDM Higgs potential are easily described. Here we will show that this formalism also gives insight into the symmetries of the THDM. We are especially interested in symmetries which are not violated by the rge's. To this aim let us start with a translation of the rge's of the couplings $\lambda_{1,2,3,4,5,6,7}$ in the conventional notation of the potential (1) to the rge's of the parameters $\eta_{\alpha\beta}$. The one-loop renormalization group equations for $\lambda_{1,2,3,4,5,6,7}$ (including the $U(1)_Y$ and $SU(2)_L$ gauge interactions with couplings g_1 and g_2, respectively) are given by [19, 27, 28]:

$$8\pi^2 \frac{d\lambda_1}{dt} = 6\lambda_1^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12|\lambda_6|^2 \tag{8}$$
$$- \lambda_1 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right) + \frac{3}{8}g_1^4 + \frac{3}{4}g_1^2 g_2^2 + \frac{9}{8}g_2^4,$$

$$8\pi^2 \frac{d\lambda_2}{dt} = 6\lambda_2^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12|\lambda_7|^2 \tag{9}$$
$$- \lambda_2 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right) + \frac{3}{8}g_1^4 + \frac{3}{4}g_1^2 g_2^2 + \frac{9}{8}g_2^4,$$

$$8\pi^2 \frac{d\lambda_3}{dt} = (\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2\lambda_3^2 + \lambda_4^2 + |\lambda_5|^2 + 2|\lambda_6|^2 + 2|\lambda_7|^2 + 4\lambda_6\lambda_7^* + 4\lambda_6^*\lambda_7 \tag{10}$$
$$- \lambda_3 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right) + \frac{3}{8}g_1^4 - \frac{3}{4}g_1^2 g_2^2 + \frac{9}{8}g_2^4,$$

$$8\pi^2 \frac{d\lambda_4}{dt} = (\lambda_1 + \lambda_2)\lambda_4 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 4|\lambda_5|^2 + 5|\lambda_6|^2 + 5|\lambda_7|^2 + \lambda_6\lambda_7^* + \lambda_6^*\lambda_7 \tag{11}$$
$$- \lambda_4 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right) + \frac{3}{2}g_1^2 g_2^2,$$

$$8\pi^2 \frac{d\lambda_5}{dt} = \lambda_5 \left(\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4 \right) + 5\lambda_6^2 + 5\lambda_7^2 + 2\lambda_6\lambda_7 \tag{12}$$
$$- \lambda_5 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right),$$

$$8\pi^2 \frac{d\lambda_6}{dt} = 6\lambda_1\lambda_6 + 3\lambda_3(\lambda_6 + \lambda_7) + \lambda_4(4\lambda_6 + 2\lambda_7) + \lambda_5(5\lambda_6^* + \lambda_7^*) \tag{13}$$
$$- \lambda_6 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right),$$

$$8\pi^2 \frac{d\lambda_7}{dt} = 6\lambda_2\lambda_7 + 3\lambda_3(\lambda_6 + \lambda_7) + \lambda_4(2\lambda_6 + 4\lambda_7) + \lambda_5(\lambda_6^* + 5\lambda_7^*) \tag{14}$$
$$- \lambda_7 \left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right).$$

In terms of $\eta_{\alpha\beta}$, they become

$$8\pi^2 \frac{d\eta_{00}}{dt} = 4\eta_{00}^2 + \eta_{00}(\eta_{11} + \eta_{22} + \eta_{33}) + \eta_{11}^2 + \eta_{22}^2 + \eta_{33}^2 + 6(\eta_{01}^2 + \eta_{02}^2 + \eta_{03}^2) \tag{15}$$
$$+ 2(\eta_{12}^2 + \eta_{13}^2 + \eta_{23}^2) - \eta_{00}\left(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2\right) + \frac{3}{4}g_1^4 + \frac{9}{4}g_2^4,$$

$$8\pi^2 \frac{d\eta_{01}}{dt} = \eta_{01}\left(6\eta_{00} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + 6(\eta_{01}\eta_{11} + \eta_{02}\eta_{12} + \eta_{03}\eta_{13}), \tag{16}$$

$$8\pi^2 \frac{d\eta_{02}}{dt} = \eta_{02}\left(6\eta_{00} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + 6(\eta_{01}\eta_{12} + \eta_{02}\eta_{22} + \eta_{03}\eta_{23}), \tag{17}$$

$$8\pi^2 \frac{d\eta_{03}}{dt} = \eta_{03}\left(6\eta_{00} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + 6(\eta_{01}\eta_{13} + \eta_{02}\eta_{23} + \eta_{03}\eta_{33}), \tag{18}$$

$$8\pi^2 \frac{d\eta_{11}}{dt} = \eta_{11}\left(3\eta_{00} + 3\eta_{11} - \eta_{22} - \eta_{33} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + \frac{3}{2}g_1^2 g_2^2 \tag{19}$$
$$+ 6\eta_{01}^2 + 4(\eta_{12}^2 + \eta_{13}^2),$$

$$8\pi^2 \frac{d\eta_{22}}{dt} = \eta_{22}\left(3\eta_{00} - \eta_{11} + 3\eta_{22} - \eta_{33} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + \frac{3}{2}g_1^2 g_2^2 \tag{20}$$
$$+ 6\eta_{02}^2 + 4(\eta_{12}^2 + \eta_{23}^2),$$

$$8\pi^2 \frac{d\eta_{33}}{dt} = \eta_{33}\left(3\eta_{00} - \eta_{11} - \eta_{22} + 3\eta_{33} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + \frac{3}{2}g_1^2 g_2^2 \tag{21}$$
$$+ 6\eta_{03}^2 + 4(\eta_{13}^2 + \eta_{23}^2),$$

$$8\pi^2 \frac{d\eta_{12}}{dt} = \eta_{12}\left(3\eta_{00} + 3\eta_{11} + 3\eta_{22} - \eta_{33} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + 6\eta_{01}\eta_{02} + 4\eta_{13}\eta_{23}, \tag{22}$$

$$8\pi^2 \frac{d\eta_{13}}{dt} = \eta_{13}\left(3\eta_{00} + 3\eta_{11} - \eta_{22} + 3\eta_{33} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + 6\eta_{01}\eta_{03} + 4\eta_{12}\eta_{23}, \tag{23}$$

$$8\pi^2 \frac{d\eta_{23}}{dt} = \eta_{23}\left(3\eta_{00} - \eta_{11} + 3\eta_{22} + 3\eta_{33} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right) + 6\eta_{02}\eta_{03} + 4\eta_{12}\eta_{13}. \tag{24}$$

We now look for symmetries among the couplings $\eta_{\alpha\beta}$ which are preserved by the rge's. Whereas (8) to (14) are not particularly illuminating, (15) to (24) tell us immediately that the three conditions

$$\eta_{01} = \eta_{02} = \eta_{03}, \quad \eta_{11} = \eta_{22} = \eta_{33}, \quad \eta_{12} = \eta_{13} = \eta_{23}, \tag{25}$$

are preserved by the rge's. Even though the rge's in general mix the quartic couplings, these conditions are maintained by them. The corresponding quartic part of the potential is given by

$$V_4^{\text{symb 5)}} = \eta_{00}K_0^2 + \eta_{11}(K_1^2 + K_2^2 + K_3^2) + 2\eta_{12}(K_1 K_2 + K_1 K_3 + K_2 K_3) + 2\eta_{01}K_0(K_1 + K_2 + K_3). \tag{26}$$

With respect to the classification which we introduce later, this quartic part of the Higgs potential is denoted as case 5). This quartic potential has the apparent symmetry S_3, generated by $K_1 \to K_2 \to K_3 \to K_1$ and $K_1 \to K_2 \to K_1$. If $\eta_{01} = 0$, denoted as case 9), then the symmetry is $S_3 \times Z_2$ from having in addition the transformation $K_{1,2,3} \to -K_{1,2,3}$. If $\eta_{12} = 0$ as well, denoted as case 12), then the symmetry is $O(3)$. These three unequal symmetries coexist in $V_4^{\text{symb 5)}}$ even after renormalization. This is our first example of the phenomenon of *symbiotic symmetries*. In terms of $\Phi_{1,2}$, the respective symmetries are Z_6 (generated by $e = \frac{1}{2}(\mathbf{1}_2 - i[\sigma_1 + \sigma_2 + \sigma_3])$), Q_{12} (generated by e and $c_3 = (i/\sqrt{2})(\sigma_1 - \sigma_2)$), and $SU(2)$. Case 12) with the symmetry $SU(2)$ was already known more than 30 years ago [3]. We now recognize the other possible symmetries with nonzero $\lambda_{6,7}$ for the first time. They are likely to remain hidden if not for the Pauli formalism.

The quadratic part of the Higgs potential complementing the quartic part (26) is given by

$$V_2^{\text{symb 5)}} = \xi_0 K_0 + \xi_1(K_1 + K_2 + K_3) \tag{27}$$

Note that the rge's of the couplings $\eta_{\alpha\beta}$, (15) to (24), depend only on the quartic parameters themselves and not on the quadratic ξ_α parameters. Therefore any THDM with the quartic part V_4 given by (26) and arbitrary quadratic

parameters ξ_α will maintain its symbiotic symmetries. In the conventional notation, the conditions (25) read

$$\operatorname{Im} m_{12}^2 = -\operatorname{Re} m_{12}^2 = \frac{1}{2}(m_{11}^2 - m_{22}^2), \quad 2\lambda_4 = \lambda_1 + \lambda_2 - 2\lambda_3, \quad \operatorname{Re}(\lambda_5) = 0,$$

$$\operatorname{Re}(\lambda_7) - \operatorname{Re}(\lambda_6) = \operatorname{Im}(\lambda_5), \quad \operatorname{Re}(\lambda_7) + \operatorname{Re}(\lambda_6) = \frac{1}{2}(\lambda_1 - \lambda_2), \quad \operatorname{Im}(\lambda_6) = -\operatorname{Re}(\lambda_6), \quad \operatorname{Im}(\lambda_7) = -\operatorname{Re}(\lambda_7), \tag{28}$$

so that this potential with all its symmetries is of the form

$$
\begin{aligned}
V^{\text{symb 5)}} = \ & m_{11}^2 \left[(\Phi_1^\dagger \Phi_1) + \operatorname{Re}(\Phi_1^\dagger \Phi_2) + \operatorname{Im}(\Phi_1^\dagger \Phi_2) \right] \\
& + m_{22}^2 \left[(\Phi_2^\dagger \Phi_2) - \operatorname{Re}(\Phi_1^\dagger \Phi_2) - \operatorname{Im}(\Phi_1^\dagger \Phi_2) \right] \\
& + \frac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) \\
& + \frac{1}{2}(\lambda_1 + \lambda_2 - 2\lambda_3)(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) - \operatorname{Im}(\lambda_5)\operatorname{Im}(\Phi_1^\dagger \Phi_2)^2 \\
& + \left[\frac{1}{2}(\lambda_1 - \lambda_2) - \operatorname{Im}(\lambda_5) \right] (\Phi_1^\dagger \Phi_1)[\operatorname{Re}(\Phi_1^\dagger \Phi_2) + \operatorname{Im}(\Phi_1^\dagger \Phi_2)] \\
& + \left[\frac{1}{2}(\lambda_1 - \lambda_2) + \operatorname{Im}(\lambda_5) \right] (\Phi_2^\dagger \Phi_2)[\operatorname{Re}(\Phi_1^\dagger \Phi_2) + \operatorname{Im}(\Phi_1^\dagger \Phi_2)].
\end{aligned}
\tag{29}
$$

As it appears, the underlying symmetries of this potential are far from being obvious. We have thus demonstrated the utility of the Pauli formalism, and its use in all future studies of the THDM is advised.

Note that this potential is CP conserving. This follows from the sufficient condition that all parameters in the potential are real, or it can be inferred from the necessary and sufficient conditions in [16]. Hence this model has five Higgs bosons with definite CP properties. There are two charged Higgs bosons $H^\pm$, two CP even Higgs bosons h^0 and H^0, as well as one CP odd pseudoscalar Higgs boson A^0. The conditions for stability and electroweak symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$ are derived in a straightforward manner using the methods described in [12]. Here, stability in the *strong* sense (i.e. guaranteed by the quartic terms) requires both conditions (30) and (31) to be fulfilled:

$$\eta_{00} + \eta_{11} + 2\eta_{12} > 2\sqrt{3}\left|\eta_{01}\right| \tag{30}$$

$$\frac{3\eta_{01}^2}{(\eta_{11} + 2\eta_{12})^2} \geq 1 \quad \text{or} \quad \frac{3\eta_{01}^2}{\eta_{11} + 2\eta_{12}} < \eta_{00} \tag{31}$$

For the model to have spontaneous symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$, we also require the condition

$$\xi_0 < \sqrt{3}\left|\xi_1\right|. \tag{32}$$

In the same way, the global minimum of the Higgs potential can be analytically obtained among the stationary solutions. Under the assumption that we have chosen parameter values such that the potential is stable and has the required electroweak symmetry breaking behavior, the masses of the neutral physical Higgs bosons are

$$
\begin{aligned}
m_{A^0} &= m_{H^\pm}^2 + 2v^2(\eta_{11} - \eta_{12}), \\
m_{h/H}^2 &= \frac{1}{2}m_{H^\pm}^2 + v^2(\eta_{11} + \eta_{12}) - \xi_0 - \xi_1 \\
& \mp \sqrt{\frac{1}{4}(m_{H^\pm}^2 + 2(v^2(\eta_{11} + \eta_{12}) + \xi_0))^2 + \xi_1(m_{H^\pm}^2 + 2(v^2(\eta_{11} + \eta_{12}) + \xi_0)) + 9\xi_1^2}
\end{aligned}
\tag{33}
$$

with $v \simeq 246$ GeV, being the SM vacuum expectation value. The charged Higgs-boson mass $m_{H^\pm}$ follows directly from the stationarity conditions.

Now let us consider more details regarding (25). Examination of (16) to (18) shows that the additional condition $\eta_{01} = \eta_{02} = \eta_{03} = 0$ is also preserved by the rge's. This corresponds to

$$\lambda_1 = \lambda_2 \tag{34}$$

in addition to the conditions (28). Using (8) and (9) with (28), we find

$$8\pi^2 \frac{d}{dt}(\lambda_1 - \lambda_2) = (\lambda_1 - \lambda_2)\left(6\lambda_1 + 6\lambda_2 - 12\,\mathrm{Im}(\lambda_5) - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right), \tag{35}$$

showing that indeed $\lambda_1 = \lambda_2$ is a solution, i.e. preserved by the rge's, even in the presence of the gauge couplings $g_{1,2}$. In addition, (12) to (14) reduce to just one equation, i.e.

$$8\pi^2 \frac{d\,\mathrm{Im}(\lambda_5)}{dt} = \mathrm{Im}(\lambda_5)\left(8\lambda_1 - 2\lambda_3 - 6\,\mathrm{Im}(\lambda_5) - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right). \tag{36}$$

As pointed out already, the resulting symmetry is Q_{12} with character table given below.

n	h	1^{++}	1^{+-}	1^{-+}	1^{--}	2^+	2^-
1	1	1	1	1	1	2	2
1	2	1	-1	-1	1	2	-2
2	3	1	1	1	1	-1	-1
2	6	1	-1	-1	1	-1	1
3	4	1	1	-1	-1	0	0
3	4	1	-1	1	-1	0	0

TABLE I: Character table of Q_{12}.

Examination of (22) to (24) shows that the additional conditions $\eta_{01} = \eta_{02} = \eta_{03} = 0$, $\eta_{12} = \eta_{13} = \eta_{23} = 0$ are also preserved by the rge's. This case corresponds to $\mathrm{Im}(\lambda_5) = 0$ in addition to (28) and (34). It is obviously supported by (36), as discussed already in [3]. On the other hand, we see from (19) to (21) that the additional condition $\eta_{11} = \eta_{22} = \eta_{33} = 0$ is *not* preserved by the rge's, which would have resulted in the symmetry $O(8)$. Altogether we have found three related models, i.e. the symbiotic model corresponding to the conditions (25) followed by the models with the additional conditions $\eta_{01} = 0$ and $\eta_{01} = \eta_{12} = 0$, respectively. These models, i.e. cases 5), 9), and (12), belong to one class of symbiotic models, denoted by class I) below.

We now present the complete set of models with a symmetry consistent with the transparent rge's (15) to (24). The symmetries found are summarized in Table II. In this table, the cases 5), 9), and 12) are discussed already. In the columns 2 to 10 the conditions among the couplings $\eta_{\alpha\beta}$ are given. In the column denoted by 'invariant terms', the allowed potential terms respecting the conditions are shown explicitly. The last two columns then give the symmetries of the potential in addition to $SU(2)_L \times U(1)_Y$. The next-to-last column gives the symmetry in terms of the $(\Phi_1, \Phi_2)^{\mathrm{T}}$ basis. The cases 6) to 9) have a quaternion symmetry in this basis. The quaternion groups Q_{4n} have $4n$ elements. For example, Q_8 consists of $\pm 1, \pm a_{1,2,3}$ or $\pm 1, \pm a_3, \pm b_3, \pm c_3$, and Q_{12} consists of $\pm 1, \pm c_{1,2,3}, \pm e, \pm e^2$ or $\pm 1, \pm b_{1,2}, \pm c_3, \pm d_3, \pm d_3^2$. The last column gives the symmetry in terms of the $\mathbf{K}$ basis. The cases 5), 6), 9), and 12) have no variations. All other cases have 3 variations each. Thus from the table we see that we have in total 28 models. In the $\Phi_{1,2}$ basis, there are several basic transformations which correspond to $K_i \to \pm K_j$. Note that $\Phi_1 \to \Phi_2^\dagger$ (which would have meant $K_{1,3} \to K_{1,3}$, $K_2 \to -K_2$) is not allowed, and similarly for any transformation with determinant -1. We have suppressed such cases.

case	η_{01}	η_{02}	η_{03}	η_{12}	η_{13}	η_{23}	η_{11}	η_{22}	η_{33}	invariant terms	symmetries $(\Phi_1,\Phi_2)^{\mathrm T}$	symmetries $\mathbf{K}$
1)	0	0	$\surd$	$\surd$	0	0	$\surd$	$\surd$	$\surd$	$K_3, K_1K_2, K_1^2, K_2^2, K_3^2$	$a_3=i\sigma_3\in Z_4$	$A_3=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\in Z_2$
2)	$\surd$	η_{01}	0	$\surd$	$\surd$	$-\eta_{13}$	$\surd$	η_{11}	$\surd$	$K_1+K_2, K_1K_2,$ $(K_1-K_2)K_3,$ $K_1^2+K_2^2, K_3^2$	$b_3=\frac{i}{\sqrt2}(\sigma_1+\sigma_2)$ $\in Z_4$	$B_3=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}\in Z_2$
3)	$\surd$	$-\eta_{01}$	0	$\surd$	$\surd$	η_{13}	$\surd$	η_{11}	$\surd$	$K_1-K_2, K_1K_2,$ $(K_1+K_2)K_3, K_3^2,$ $K_1^2+K_2^2$	$c_3=\frac{i}{\sqrt2}(\sigma_1-\sigma_2)$ $\in Z_4$	$C_3=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}\in Z_2$
4)	$\surd$	η_{01}	$-\eta_{01}$	$\surd$	$-\eta_{12}$	$-\eta_{12}$	$\surd$	η_{11}	η_{11}	$K_1+K_2-K_3,$ $K_1K_2-(K_1+K_2)K_3,$ $K_1^2+K_2^2+K_3^2$	$d_3=$ $\frac12(1+i[\sigma_1+\sigma_2-\sigma_3])$ $\in Z_6$	$D_3=\begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix}\in S_3$
5)	$\surd$	η_{01}	η_{01}	$\surd$	η_{12}	η_{12}	$\surd$	η_{11}	η_{11}	$K_1+K_2+K_3,$ $K_1K_2+K_1K_3+K_2K_3,$ $K_1^2+K_2^2+K_3^2$	$e=$ $\frac12(1-i[\sigma_1+\sigma_2+\sigma_3])$ $\in Z_6$	$E=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\in S_3$
6)	0	0	0	0	0	0	$\surd$	$\surd$	$\surd$	K_1^2, K_2^2, K_3^2	$a_{1,2}=i\sigma_{1,2}\in Q_8$	$A_{1,2}\in Z_2\times Z_2\times Z_2$
7)	0	0	0	$\surd$	0	0	$\surd$	η_{11}	$\surd$	$K_1K_2, K_1^2+K_2^2, K_3^2$	$a_3,b_3\in Q_8$	$A_3,B_3\in Z_2\times Z_2$
8)	0	0	0	$\surd$	$-\eta_{12}$	$-\eta_{12}$	$\surd$	η_{11}	η_{11}	$K_1K_2-(K_1+K_2)K_3,$ $K_1^2+K_2^2+K_3^2$	$d_3,b_1\in Q_{12}$	$D_3,B_1\in S_3\times Z_2$
9)	0	0	0	$\surd$	η_{12}	η_{12}	$\surd$	η_{11}	η_{11}	$K_1K_2+K_1K_3+K_2K_3,$ $K_1^2+K_2^2+K_3^2$	$e,c_3\in Q_{12}$	$E,C_3\in S_3\times Z_2$
10)	0	0	$\surd$	0	0	0	$\surd$	η_{11}	$\surd$	$K_3, K_1^2+K_2^2, K_3^2$	$r_3(\theta)=$ $\cos(\frac{\theta}{2})+i\sin(\frac{\theta}{2})\sigma_3$ $\in U(1)$	$R_3(\theta)=\begin{pmatrix}c_\theta&-s_\theta&0\\s_\theta&c_\theta&0\\0&0&1\end{pmatrix}\in O(2)$
11)	0	0	0	0	0	0	$\surd$	η_{11}	$\surd$	$K_1^2+K_2^2, K_3^2$	$r_3(\theta),b_3\in U(1)\times Z_2$	$R_3(\theta),B_3\in O(2)\times Z_2$
12)	0	0	0	0	0	0	$\surd$	η_{11}	η_{11}	$K_1^2+K_2^2+K_3^2$	$SU(2)$	$O(3)$

TABLE II: *Symmetries preserved by the rge's. In the first nine columns following the case number, the conditions with respect to the couplings $\eta_{\alpha\beta}$ are given. Explicitly, the corresponding invariant terms of the THDM Higgs potential are shown. The next-to-last column gives the corresponding symmetry of the potential in the $(\Phi_1,\Phi_2)^{\mathrm T}$ basis, whereas the last column gives the symmetry in terms of the $\mathbf{K}$ basis. In case 10) in the last column we use the abbreviations $c_\theta=\cos(\theta)$ and $s_\theta=\sin(\theta)$.*

We may also classify the various cases of symbiotic symmetries. This was discussed already for the case 5) where we have subcases 9) and 12) with additional conditions. In this sense, we group the related models 5), 9), and 12) in one class of models. In an analogous way we find the classes of related cases as given in Table III. In addition, we give in this table for each class the symmetry of the models with respect to the $(\Phi_1,\Phi_2)^{\mathrm T}$ basis. Here, we have assumed that this basis is determined outside the Higgs potential, i.e. by their Yukawa couplings, which will of course break the symmetries we have discussed in this paper. However, except for the couplings proportional to m_t, they are small compared to the gauge couplings. If all Yukawa couplings are neglected, then class II) in the above is equivalent to class I) because $b_3^\dagger e b_3 = d_3^\dagger$, and class V) is equivalent to class IV) because $a_1^\dagger b_3 a_1 = c_3$.

class	related cases	symmetries $(\Phi_1, \Phi_2)^{\mathrm{T}}$
I)	5) $\to$ 9) $\to$ 12):	$Z_6 \to Q_{12} \to SU(2)$
II)	4) $\to$ 8) $\to$ 12):	$Z_6 \to Q_{12} \to SU(2)$ [3 variations]
III)	10) $\to$ 11):	$U(1) \to U(1) \times Z_2$ [3 variations]
IV)	2) $\to$ 7) $\to$ 11):	$Z_4 \to Q_8 \to U(1) \times Z_2$ [3 variations]
V)	3) $\to$ 7) $\to$ 11):	$Z_4 \to Q_8 \to U(1) \times Z_2$ [3 variations]
VI)	1) $\to$ 6):	$Z_4 \to Q_8$ [3 variations]

TABLE III: *Classes of symbiotic symmetries for all related cases shown in Table II. The subsequent cases originate from the previous case by an additional condition. Also given are the symmetries with respect to the $(\Phi_1, \Phi_2)^{\mathrm{T}}$ basis and the number of variations.*

Let us summarize our findings. Recently it was shown that by using Pauli matrices, the formalism of gauge-invariant functions simplifies the study of THDM's. We have determined the renormalization-group equations of the parameters $\eta_{\alpha\beta}$ of the Higgs potential in this approach. In so doing, relations among these couplings become completely transparent, allowing us to find all possible symmetries which are preserved by the rge's. We discover cases where the quartic Higgs potential has two or more separate sectors with unequal symmetries, but are nevertheless maintained by the rge's, including the gauge interactions. We call this the phenomenon of *symbiotic symmetries*. In a systematic way, we have obtained all possible models with a symmetry beyond that of the SM, as shown in Table II. There are 12 basic scenarios, 8 of which have 3 variations, for a total of 28 such models. There are 6 symbiotic classes as shown in Table III. These symmetries are very much hidden in the $\lambda_{1,2,3,4,5,6,7}$ parameterization of the Higgs potential, but become totally transparent in the Pauli formalism.

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